Math 1010 I Week 1

Set notation

- A set is a collection of elements

eg. $A = \{2, 4, 6, 8\}$

set: A — elements: $2, 4, 6, 8$

i.e. A is the set of the first four positive even numbers

- $x \in A$: x is an element of A

  $x \notin A$: x is not an element of A

eg. $2 \in \{2,4,6,8\}$

$3 \notin \{2,4,6,8\}$

- $A \subseteq B$: A is a subset of B

  i.e. every element of A is an element of B

  $\{2,4\} \subseteq \{2,4,6,8\} \subseteq \{2,4,6,8,10\}$

Rmk

(1) order is not important $\{2,5,7\} = \{5,2,7\}$

(2) Many possible presentation for a set

i). $\underbrace{\{x \in \mathbb{R} \mid x^2 = 1\}}_{\text{the set of all real numbers } x \text{ such that } x^2 = 1} = \{1, -1\}$

such that

ii). $\underbrace{\{2m : m \in \mathbb{N}\}}_{\text{the set of all } 2m \text{ such that } m \text{ is a natural number}} = \{2, 4, 6, 8, 10, \ldots\}$

such that

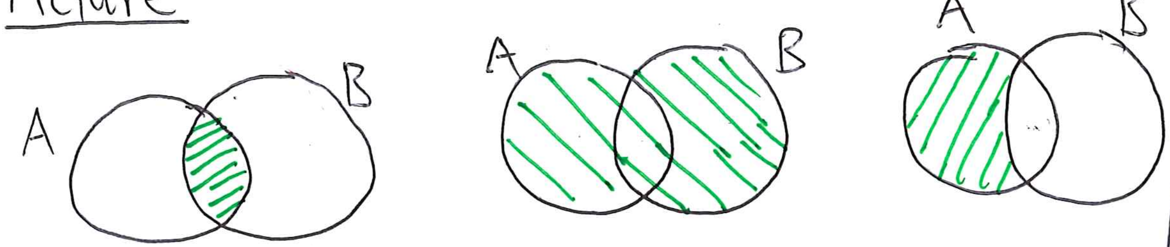
Notations: Let A, B be sets

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ (intersection)

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (union)

$A \setminus B = \{x \in A \mid x \notin B\}$ (Relative complement of B in A)

Picture



$A \cap B$ $\quad$ $A \cup B$ $\quad$ $A \setminus B$

eg. $A = \{2, 4, 6\}$ $\quad$ $B = \{3, 6, 9\}$

$A \cup B = \{2, 3, 4, 6, 9\}$

$A \cap B = \{6\}$

$A \setminus B = \{2, 4\}$

Some important sets

$\mathbb{N}$ = the set of all natural numbers

$= \{1, 2, 3, 4, \cdots\}$

$\mathbb{Z}$ = the set of all integers

$= \{0, \pm 1, \pm 2, \pm 3, \cdots\}$

$= \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$

$\mathbb{Q}$ = the set of all rational numbers

$\mathbb{R}$ = the set of all real numbers

ϕ = empty set

$$\phi \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Intervals

Let $a, b \in \mathbb{R}$

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$

$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$

$(a, \infty) = \{x \in \mathbb{R} : x > a\}$

$(-\infty, \infty) = \mathbb{R}$

eg. $2 \in [2,4]$ ← closed interval (end points included)

$2 \notin (2,4)$ ← open interval (end points not included)

Logic symbols

$\forall$: for all / for any

$\exists$: there exists

$\exists!$: there exists unique

$\Rightarrow$: implies

$\Leftrightarrow$: if and only if (equivalent)

eg ① $\forall x \in \mathbb{R}, \; x^2 \geq 0$

② $\forall x \in \mathbb{Z}, \; \exists y \in \mathbb{Z}$ such that $y > x$.

③ $\forall x > 0, \; \exists! \, y > 0$ such that $y^2 = x$

④ Let $m \in \mathbb{Z}$

m is divisible by 4 $\Rightarrow$ m is divisible by 2

False ~~m is divisible by 2 $\Rightarrow$ m is divisible by 4~~

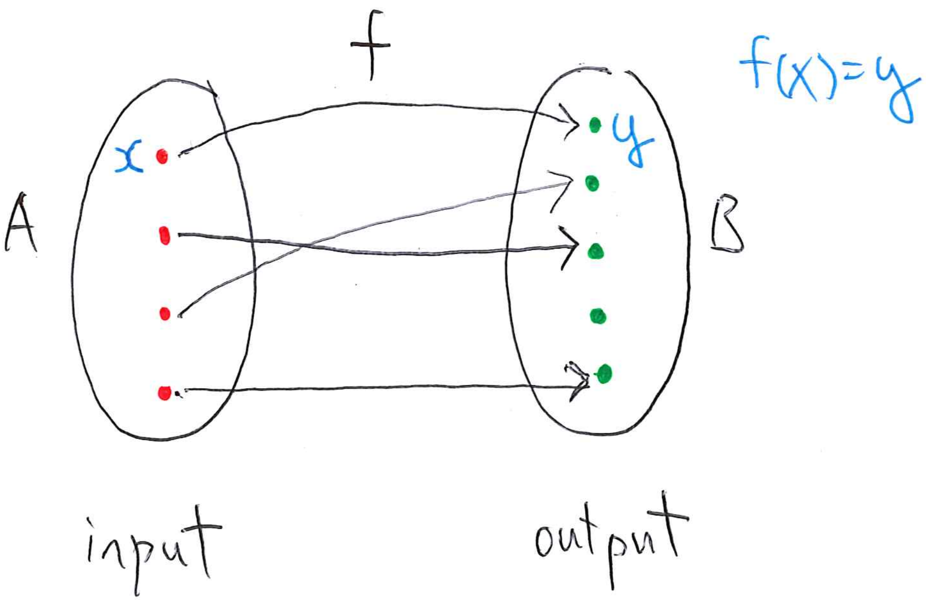
m is divisible by 6 $\Leftrightarrow$ m is divisible by 2 and 3

Function Let A, B be sets.

A function $f: A \to B$ is a rule of assigning to each element of A an element of B

A is called the domain of f

B is called the codomain of f



We say that

the image of x (under f) is y

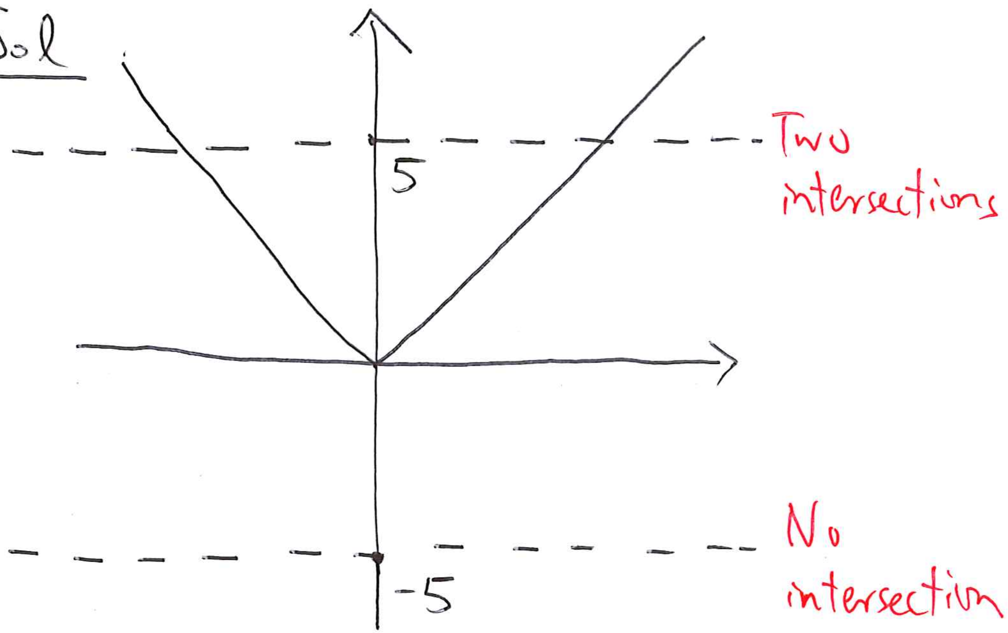
x is a pre-image of y

$$\text{range of } f = f(A) = \{f(x) \in B : x \in A\} = \{z \in B : z = f(x) \text{ for some } x \in A\}$$

e.g. $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$.

Is $5 \in f(\mathbb{R})$? Is $-5 \in f(\mathbb{R})$?

Sol



$$f(5) = f(-5) = 5$$

$$\Rightarrow 5 \in f(\mathbb{R})$$

However, $f(x) \neq -5 \quad \forall x \in \mathbb{R}$

$$\Rightarrow -5 \notin f(\mathbb{R})$$

Rmk $\underset{\substack{\shortparallel \\ \text{range of } f}}{f(\mathbb{R})} = \{x \in \mathbb{R} : x \geqslant 0\}$

Injective, surjective and bijective functions

Let $f : A \to B$ be a function.

f is said to be

① injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

i.e. "Same output $\Rightarrow$ Same input"

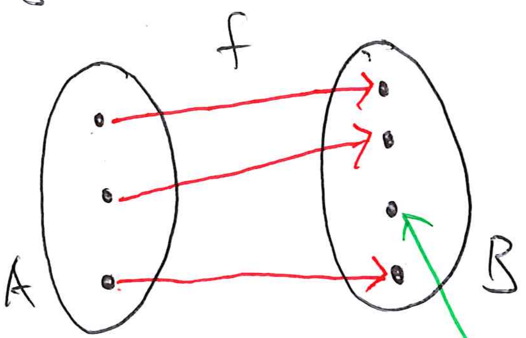
i.e. "Different inputs $\Rightarrow$ Different outputs"

② surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$

③ Bijective if f is both injective and surjective

Picture $f: A \to B$

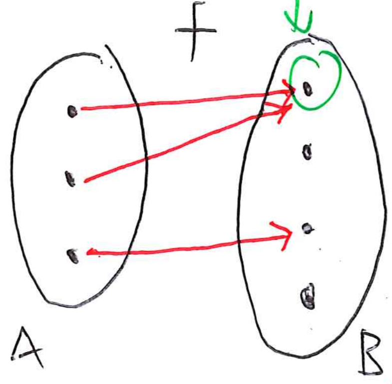
eg1



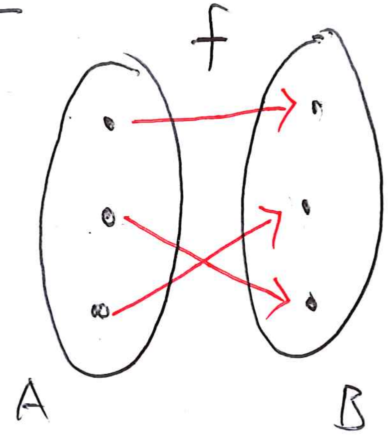
no preimage

eg2

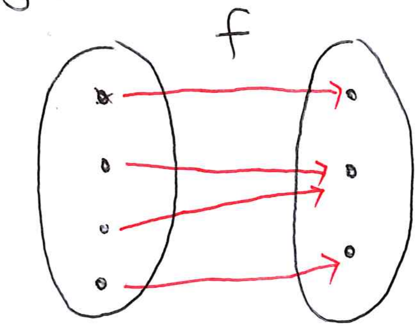
Two different elements map to it



eg3



eg4

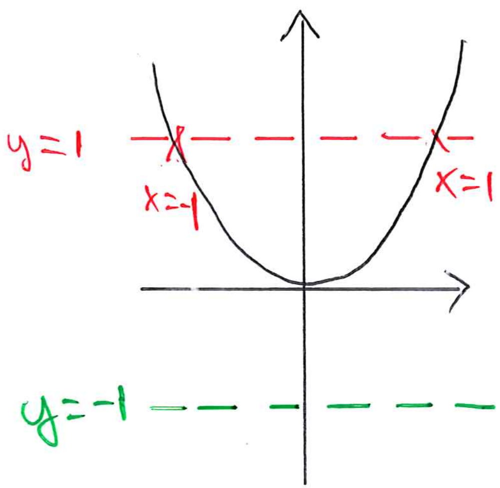


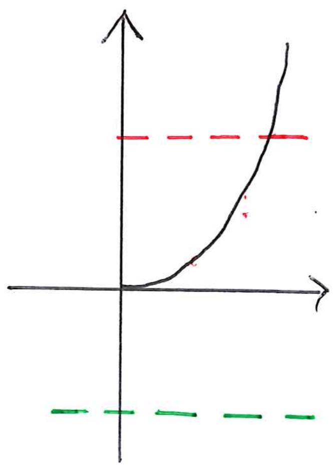
	eg1	eg2	eg3	eg4
injective	✓	✗	✓	✗
surjective	✗	✗	✓	✓
bijective	✗	✗	✓	✗

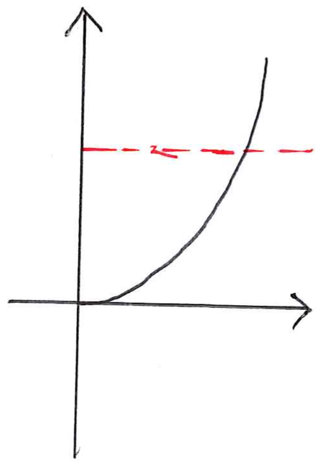
Rmk Graphically, injective means no two different arrows point to the same element in B

surjective means every element in B is pointed by at least one arrow eg Is the square function bijective?

	$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$	$g : [0,\infty) \to \mathbb{R}$, $g(x) = x^2$	$h : [0,\infty) \to [0,\infty)$, $h(x) = x^2$	$p\ [0,2] \to [0,3]$, $p(x) = x^2$
Injective?	✗	✓	✓	It is not well-defined because $2 \in [0,2]$, $2^2 = 4 \notin [0,3]$
Surjective?	✗	✗	✓	
Bijective?	✗	✗	✓	







Sequence of Real numbers

A sequence $\{a_n\}$ consists of

$a_1, a_2, a_3, a_4, \cdots\cdots$

where each $a_i \in \mathbb{R}$

Equivalently, a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$

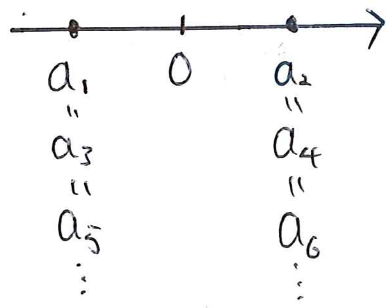
eg 1 $\quad a_n = (-1)^n$

$a_1 = a_3 = a_5 = \cdots = -1$

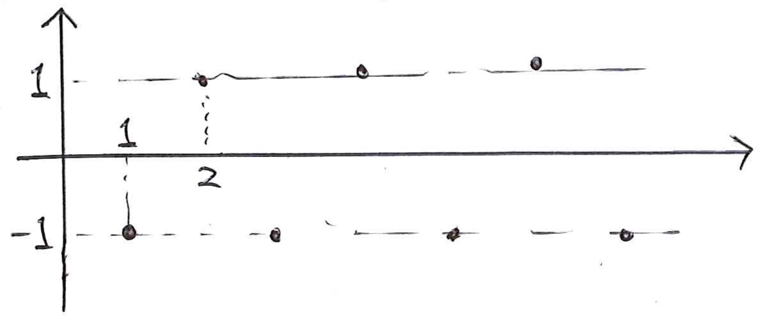
$a_2 = a_4 = a_6 = \cdots = 1$

$-1, 1, -1, 1, -1, \cdots$

Picture:



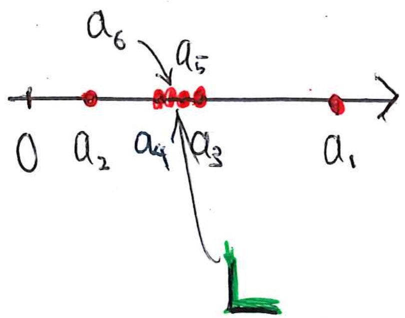
or



eg 2 (Recursive Sequence)

$a_1 = 2, \quad a_n = \dfrac{1}{1 + a_{n-1}}$ for $n \geq 2$

$\Rightarrow a_2 = \dfrac{1}{3}, \; a_3 = \dfrac{3}{4}, \; a_4 = \dfrac{4}{7}, \; a_5 = \dfrac{7}{11}$



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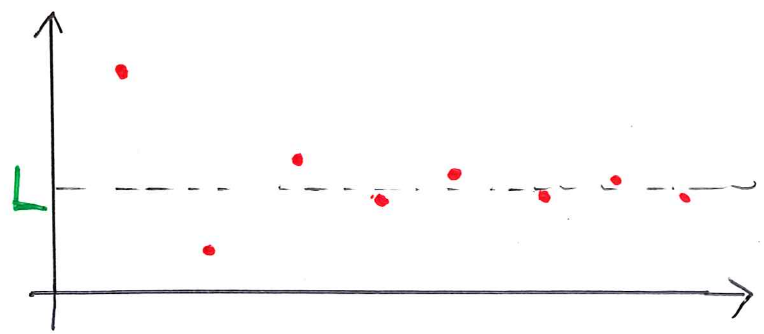


① $L = \dfrac{\sqrt{5} - 1}{2}$ ② $a_n \to L$ as $n \to \infty$

Limit of a sequence

"Defn" Let $\{a_n\}$ be a sequence, $L \in \mathbb{R}$

We say that $\boxed{\lim_{n\to\infty} a_n = L}$ if

a_n is close enough to L

when n is large enough

In this case, $\{a_n\}$ is said to be

$\boxed{\text{convergent}}$

If no such L exists, then we say

that $\boxed{\lim_{n\to\infty} a_n \text{ does not exist (DNE)}}$

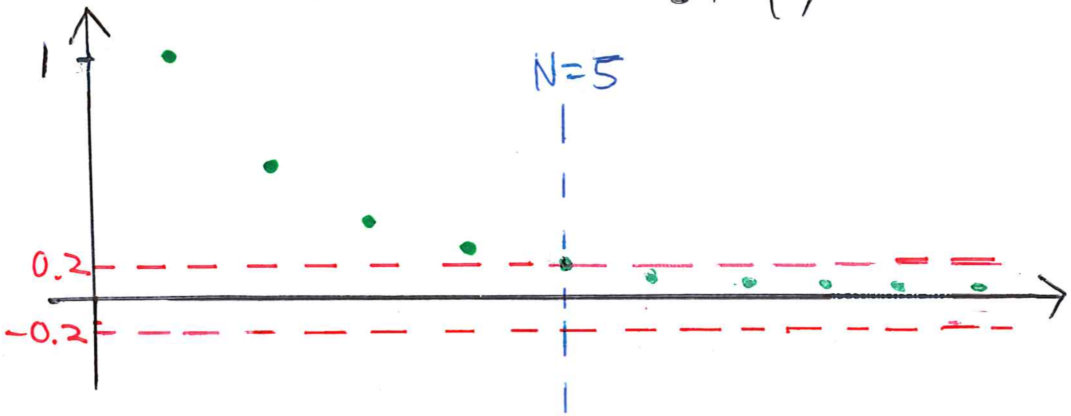
and $\{a_n\}$ is said to be $\boxed{\text{divergent}}$

Real definition of limit (ε-N)

epsilon

$\lim_{n\to\infty} a_n = L$ if $\boxed{\forall \varepsilon > 0, \exists N > 0 \text{ such that } \forall n > N, |a_n - L| < \varepsilon}$

eg 3 $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$



$\lim_{n\to\infty} a_n = 0$

eg. Given $\varepsilon = 0.2$, take $N = 5$

Given $\varepsilon = 0.01$, take $N = 100$

In general, given $\varepsilon > 0$, take N to be any integer greater than $\frac{1}{\varepsilon}$